5

Complex Numbers and Quadratic Equations

Short Answer Type Questions

Q. 1 For a positive integer n, find the value of $(1-i)^n\left(1-\frac{1}{i}\right)^n$.

Sol. Given expression $= (1-i)^n\left(1-\frac{1}{i}\right)^n$

$$= (1-i)^n(i-1)^n \cdot i^{-n} = (1-i)^n(1-i)^n(-1)^n \cdot i^{-n}$$

$$= [(1-i)^2]^n(-1)^n \cdot i^{-n} = (1+i^2-2i)^n(-1)^n i^{-n} \qquad [\because i^2 = -1]$$

$$= (1-1-2i)^n(-1)^n i^{-n} = (-2)^n \cdot i^n(-1)^n i^{-n}$$

$$= (-1)^{2n} \cdot 2^n = 2^n$$

Q. 2 Evaluate $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in N$.

Thinking Process

Use $i^2 = -1, i^4 = (-1)^2 = 1, i^3 = -i$, and $i^5 = i$ to solve it

Sol. Given that, $\sum_{n=1}^{13}(i^n + i^{n+1}), n \in N$

$$= (i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13})$$
$$+ (i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13} + i^{14})$$

$$= (i + 2i^2 + 2i^3 + 2i^4 + 2i^5 + 2i^6 + 2i^7 + 2i^8 + 2i^9 + 2i^{10} + 2i^{11} + 2i^{12} + 2i^{13} + i^{14})$$

$$= i - 2 - 2i + 2 + 2i + 2(i^4)i^2 + 2(i)^4 i^3 + 2(i^2)^4 + 2(i^2)^4 i + 2(i^2)^5$$
$$+ 2(i^2)^5 \cdot i + 2(i^2)^6 + 2(i^2)^6 \cdot i + (i^2)^7$$

$$= i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 1 - 1 + i$$

Alternate Method

$$\sum_{n=1}^{13}(i^n + i^{n+1}),\ n \in N = \sum_{h=1}^{13} i^n(1+i)$$

$$= (1+i)[i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}]$$

$$= (1+i)[i^{13}] \quad [\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{12} i^n = 0$$

$$= (1+i)i$$

$[\because (i^4)^3 \cdot i = i]$

$$= (i^2 + i) = i - 1$$

Q. 3 If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

Thinking Process

If two complex numbers $z_1 = x_1 + iy^1$ and $z_2 = x_2 + iy_2$ are equal

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.

Sol. Given that, $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$ ...(i)

$$\therefore \quad \left(\frac{1+i}{1-i}\right)^3 = \frac{1+i^3+3i(1+i)}{1-i^3-3i(1-i)} = \frac{1-i+3i+3i^2}{1+i-3i+3i^2}$$

$$= \frac{2i-2}{-2i-2} = \frac{i-1}{-i-1} = \frac{1-i}{1+i}$$

$$= \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1+i^2-2i}{1+1} = \frac{1-1-2i}{2}$$

$$\Rightarrow \quad \left(\frac{1+i}{1-i}\right)^3 = -i \quad \text{...(ii)}$$

Similarly, $\left(\frac{1-i}{1+i}\right)^3 = \frac{-1}{i} = \frac{i^2}{i} = i$ ...(iii)

Using Eqs. (ii) and (iii) in Eq. (i), we get

$$-i - i = x + iy$$

$$\Rightarrow \quad -2i = x + iy$$

On comparing real and imaginary part of complex number, we get

$$x = 0 \text{ and } y = -2$$

So, $(x, y) = (0, -2)$

Q. 4 If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Sol. Given that, $\frac{(1+i)^2}{2-i} = x + iy$

$$\Rightarrow \quad \frac{(1+i^2+2i)}{2-i} = x + iy \Rightarrow \frac{2i}{2-i} = x + iy$$

$$\Rightarrow \quad \frac{2i(2+i)}{(2-i)(2+i)} = x + iy \Rightarrow \frac{4i+2i^2}{4-i^2} = x + iy$$

$$\Rightarrow \quad \frac{4i-2}{4+1} = x + iy \Rightarrow \frac{-2}{5} + \frac{4i}{5} = x + iy$$

On comparing both sides, we get

$$x = -2/5 \Rightarrow y = 4/5$$

$$\Rightarrow \quad x + y = \frac{-2}{5} + \frac{4}{5} = 2/5$$

Q. 5 If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Sol. Given that, $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$

$$\Rightarrow \quad \left[\frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}\right]^{100} = a + ib \Rightarrow \left(\frac{1+i^2-2i}{1-i^2}\right)^{100} = a + ib$$

$$\Rightarrow \quad \left(\frac{-2i}{2}\right)^{100} = a + ib \qquad [\because i^2 = -1]$$

$$\Rightarrow \quad (i^4)^{25} = a + ib \Rightarrow 1 = a + ib$$

Then, $a = 1$ and $b = 0$ $\qquad [\because i^4 = 1]$

$$\therefore \quad (a, b) = (1, 0)$$

Q. 6 If $a = \cos\theta + i\sin\theta$, then find the value of $\frac{1+a}{1-a}$.

Thinking Process

To solve the above problem use the trigonometric formula $\cos\theta = 2\cos^2\theta/2 - 1 = 1 - 2\sin^2\theta/2$ *and* $\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$.

Sol. Given that, $a = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1+a}{1-a} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1+2\cos^2\theta/2 - 1 + 2i\sin\theta/2\cdot\cos\theta/2}{1-1+2\sin^2\theta/2 - 2i\sin\theta/2\cdot\cos\theta/2} = \frac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2\sin\theta/2(\sin\theta/2 - i\cos\theta/2)}$$

$$= -\frac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2i\sin\theta/2(\cos\theta/2 + i\sin\theta/2)} = -\frac{1}{i}\cot\theta/2$$

$$= \frac{+i^2}{i}\cot\theta/2 = i\cot\theta/2 \qquad \left[\because \frac{-1}{i} = \frac{i^2}{i}\right]$$

Q. 7 If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. We have, $(1+i)z = (1-i)\bar{z} \Rightarrow \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}\frac{(1-i)}{(1-i)} \Rightarrow \frac{z}{\bar{z}} = \frac{1+i^2-2i}{1-i^2} \qquad [\because i^2 = -1]$$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1-1-2i}{2} \Rightarrow \frac{z}{\bar{z}} = -i$$

$\therefore \quad z = -i\bar{z}$ **Hence proved.**

Q. 8 If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$, represents a circle.

Sol. Given that, $z = x + iy$

Then, $\bar{z} = x - iy$

Now, $z\bar{z} + 2(z + \bar{z}) + b = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$

$\Rightarrow x^2 + y^2 + 4x + b = 0$, which is the equation of a circle.

Q. 9 If the real part of $\dfrac{\bar{z} + 2}{\bar{z} - 1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let $z = x + iy$

Now,
$$\frac{\bar{z}+2}{\bar{z}-1} = \frac{x - iy + 2}{x - iy - 1}$$
$$= \frac{[(x+2) - iy][(x-1) + iy]}{[(x-1) - iy][(x-1) + iy]}$$
$$= \frac{(x-1)(x+2) - iy(x-1) + iy(x+2) + y^2}{(x-1)^2 + y^2}$$
$$= \frac{(x-1)(x+2) + y^2 + i[(x+2)y - (x-1)y]}{(x-1)^2 + y^2} \quad [\because -i^2 = 1]$$

Taking real part, $\dfrac{(x-1)(x+2) + y^2}{(x-1)^2 + y^2} = 4$

$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$

$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0$, which represents a circle.

Hence, z lies on the circle.

Q. 10 Show that the complex number z, satisfying the condition $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ lies on a circle.

Thinking Process

First use, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. *Also apply* $\arg(z) = \theta = \tan^{-1}\dfrac{y}{x}$, *where* $z = x + iy$ *and then use the property* $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x - y}{1 + xy}\right)$

Sol. Let $z = x + iy$

Given that, $\arg\left(\dfrac{z-1}{z+1}\right) = \pi/4$

$\Rightarrow \arg(z - 1) - \arg(z + 1) = \pi/4$

$\Rightarrow \arg(x + iy - 1) - \arg(x + iy + 1) = \pi/4$

$\Rightarrow \arg(x - 1 + iy) - \arg(x + 1 + iy) = \pi/4$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \pi/4$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right] = \pi/4$$

$$\Rightarrow \quad \frac{y\left[\frac{x+1-x+1}{x^2-1}\right]}{\frac{x^2-1+y^2}{x^2-1}} = \tan \pi/4$$

$$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = 1$$

$$\Rightarrow \quad x^2+y^2-1 = 2y$$

$$\Rightarrow \quad x^2+y^2-2y-1 = 0, \text{ which represents a circle.}$$

Q. 11 Solve the equation $|z| = z + 1 + 2i$.

Sol. The given equation is $|z| = z + 1 + 2i$ ...(i)

Let $z = x + iy$

From Eq. (i), $|x + iy| = x + iy + 1 + 2i$

$$\Rightarrow \quad \sqrt{x^2+y^2} = x + iy + 1 + 2i \qquad \left[\because \ |z| + \sqrt{x^2 = y^2}\right]$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1) + i\,(y+2)$$

On squaring both sides, we get

$$x^2 + y^2 = (x+1)^2 + i^2(y+2)^2 + 2i\,(x+1)(y+2)$$

$$\Rightarrow \quad x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i\,(x+1)(y+2)$$

On comparing real and imaginary parts,

$$x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

i.e., $2y^2 = 2x - 4y - 3$ ...(ii)

and $2\,(x+1)(y+2) = 0$

$(x+1) = 0$ or $(y+2) = 0$

$$\Rightarrow \quad x = -1 \text{ or } y = -2$$

For $x = -1$, $2y^2 = -2 - 4y - 3$

$2y^2 + 4y + 5 = 0$ [using Eq. (ii)]

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{16 - 2 \times 4 \times 5}}{4}$$

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

For $y = -2$, $2(-2)^2 = 2x - 4(-2) - 3$ [usign Eq. (ii)]

$$\Rightarrow \quad 8 = 2x + 8 - 3$$

$$\Rightarrow \quad 2x = 3 \Rightarrow x = 3/2$$

$$\therefore \quad z = x + iy = 3/2 - 2i$$

Long Answer Type Questions

Q. 12 If $|z+1| = z + 2(1+i)$, then find the value of z.

Sol. Given that, $|z+1| = z + 2(1+i)$ ...(i)

$$z = x + iy$$

Then, $|x+iy+1| = x + iy + 2(1+i)$

$\Rightarrow$ $|x+1+iy| = (x+2) + i(y+2)$

$\Rightarrow$ $\sqrt{(x+1)^2 + y^2} = (x+2) + i(y+2)$

On squaring both sides, we get

$$(x+1)^2 + y^2 = (x+2)^2 + i^2(y+2)^2 + 2i(x+2)(y+2)$$

$\Rightarrow$ $x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x+2)(y+2)$

$\Rightarrow$ $x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x+2)(y+2)$

On comparing real and imaginary parts, we get

$$x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$\Rightarrow$ $2y^2 - 2x + 4y + 1 = 0$ ...(ii)

and $2(x+2)(y+2) = 0$

$\Rightarrow$ $x + 2 = 0$ or $y + 2 = 0$

$x = -2$ or $y = -2$ ...(iii)

For $x = -2$, $2y^2 + 4 + 4y + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $2y^2 + 4y + 5 = 0$

$\Rightarrow$ $16 - 4 \times 2 \times 5 < 0$

$\therefore$ Discriminant, $D = b^2 - 4ac < 0$

$\Rightarrow$ $2y^2 + 4y + 5$ has no real roots.

For $y = -2$, $2(-2)^2 - 2x + 4(-2) + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $8 - 2x - 8 + 1 = 0$

$\Rightarrow$ $x = 1/2$

$\therefore$ $z = x + iy = \dfrac{1}{2} - 2i$

Q. 13 If $\arg(z-1) = \arg(z+3i)$, then find $x - 1 : y$, where $z = x + iy$.

Sol. Given that, $\arg(z-1) = \arg(z+3i)$

and let $z = x + iy$

Now, $\arg(z-1) = \arg(z+3i)$

$\Rightarrow$ $\arg(x + iy - 1) = \arg(x + iy + 3i)$

$\Rightarrow$ $\arg(x - 1 + iy) = \arg[x + i(y+3)]$

$\Rightarrow$ $\tan^{-1}\dfrac{y}{x-1} = \tan^{-1}\dfrac{y+3}{x}$

$\Rightarrow$ $\dfrac{y}{x-1} = \dfrac{y+3}{x} \Rightarrow xy = (x-1)(y+3)$

$\Rightarrow$ $xy = xy - y + 3x - 3 \Rightarrow 3x - 3 = y$

$$\Rightarrow \quad \frac{3(x-1)}{y} = 1 \Rightarrow \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore \quad (x-1) : y = 1 : 3$$

Q. 14 Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

Thinking Process

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$, use this concept to solve the above problem. Also, we know that general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Sol. Let $z = x + iy$

Given, equation is $\left|\frac{z-2}{z-3}\right| = 2 \Rightarrow \frac{|z-2|}{|z-3|} = 2$

$$\Rightarrow \quad \frac{|x+iy-2|}{|x+iy-3|} = 2$$

$$\Rightarrow \quad |x-2+iy| = 2|x-3+iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \quad \left[\because |x+iy| = \sqrt{x^2+y^2}\right]$$

On squaring both sides, we get

$$x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \quad \ldots\text{(i)}$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow \quad 2g = \frac{-20}{3} \Rightarrow g = \frac{-10}{3}$$

and

$$2f = 0 \Rightarrow f = 0 \text{ and } c = \frac{32}{3}$$

$$\therefore \quad \text{Centre} = (-g, -f) = (10/3, 0)$$

Also,

$$\text{radius } (r) = \sqrt{(10/3)^2 + 0 - 32/3} \quad [\because r = \sqrt{g^2 + f^2 - c}]$$

$$= \frac{1}{3}\sqrt{(100-96)} = 2/3$$

Q. 15 If $\frac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $|z|$.

Thinking Process

If $z = x + iy$ is a purely imaginary number, then its real part must be equal to zero i.e., $x = 0$,

Sol. Let

$$z = x + iy$$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}, \; z \neq -1$$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{(x^2-1)+iy(x+1)-iy(x-1)-i^2y^2}{(x+1)^2-(iy)^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{(x^2-1)+y^2+i[y(x+1)-y(x-1)]}{(x+1)^2+y^2}$$

Given that, $\frac{z-1}{z+1}$ is a purely imaginary numbers.

Then, $\frac{(x^2-1)+y^2}{(x+1)^2+y^2} = 0$

$\Rightarrow \quad x^2-1+y^2 = 0 \Rightarrow x^2+y^2 = 1$

$\Rightarrow \quad \sqrt{x^2+y^2} = \sqrt{1} \Rightarrow |z| = 1$ $\quad [\because |z| = \sqrt{x^2+y^2}]$

Q. 16 z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1)+\arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers.

Given that, $|z_1| = |z_2|$

and $\arg(z_1) + \arg(z_2) = \pi$

If $|z_1| = |z_2|$

$\Rightarrow \quad r_1 = r_2$ ...(i)

and if $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad \theta_1 + \theta_2 = \pi$

$\Rightarrow \quad \theta_1 = \pi - \theta_2$

Now, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad z_1 = r_2[\cos(\pi-\theta_2) + i\sin(\pi-\theta_2)]$ $\quad [\because r_1 = r_2 \text{ and } \theta_1 = (\pi-\theta_2)]$

$\Rightarrow \quad z_1 = r_2(-\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 = -r_2(\cos\theta_2 - i\sin\theta_2)$

$\Rightarrow \quad z_1 = -[r_2(\cos\theta_2 - i\sin\theta_2)]$

$\Rightarrow \quad z_1 = -\bar{z}_2$ $\quad [\because \bar{z}_2 = r_2(\cos\theta_2 - i\sin\theta_2)]$

Q. 17 If $|z_1| = 1\ (z_1 \neq -1)$ and $z_2 = \frac{z_1-1}{z_1+1}$, then show that the real part of z_2 is zero.

Sol. Let $z_1 = x + iy$

$\Rightarrow \quad |z_1| = \sqrt{x^2+y^2} = 1$ $\quad [\because |z_1| = 1, \text{given}]$ ...(i)

Now, $z_2 = \frac{z_1-1}{z_1+1} = \frac{x+iy-1}{x+iy+1}$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{x^2-1+iy(x+1)-iy(x-1)-i^2y^2}{(x+1)^2-i^2y^2}$$

$$= \frac{x^2-1+ixy+iy-ixy+iy+y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} = \frac{1 - 1 + 2iy}{(x+1)^2 + y^2} \quad [\because x^2 + y^2 = 1]$$

$$= 0 + \frac{2yi}{(x+1)^2 + y^2}$$

Hence, the real part of z_2 is zero.

Q. 18 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

Thinking Process

First let, $z = r(\cos\theta + i\sin\theta)$, then conjugate of z i.e., $\bar{z} = r(\cos\theta - i\sin\theta)$. Use the property $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

Then, $z_2 = \bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1) = r_1[\cos(-\theta_1) + \sin(-\theta_1)]$

Also, let $z_3 = r_2(\cos\theta_2 + i\sin\theta_2)$,

Then, $z_4 = \bar{z}_3 = r_2(\cos\theta_2 - i\sin\theta_2)$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$

$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 \quad [\because \arg(z) = \theta]$$

$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Q. 19 If $|z_1| = |z_2| = \cdots = |z_n| = 1$, then show that

$$|z_1 + z_2 + z_3 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}\right|.$$

Sol. Given that, $|z_1| = |z_2| = \cdots = |z_n| = 1$

$\Rightarrow$ $|z_1|^2 = |z_2|^2 = \cdots = |z_n|^2 = 1$

$\Rightarrow$ $z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = \cdots = z_n\bar{z}_n = 1$

$\Rightarrow$ $z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} = \cdots = z_n = \frac{1}{\bar{z}_n}$

Now, $|z_1 + z_2 + z_3 + z_4 + \cdots + z_n|$

$$= \left|\frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{\bar{z}_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + \cdots + \frac{z_n\bar{z}_n}{\bar{z}_n}\right| \quad \left[\because z_1\bar{z} = 1, \text{ where } z_1 = \frac{1}{z}, z_1 = \frac{\bar{z}}{\bar{z} - \bar{z}}, z_1 = \bar{z}\right]$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + \cdots + \frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \cdots + \frac{1}{\bar{z}_n}\right| = \sqrt{\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}}$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$$

Hence proved.

Q. 20 If the complex numbers z_1 and z_2, $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Given that, $\arg(z_1) - \arg(z_2) = 0$

$$\theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$$

$$z_2 = r_2(\cos\theta_1 + i\sin\theta_1) \quad [\because \theta_1 = \theta_2]$$

$$z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_1) + i(r_1\sin\theta_1 - r_2\sin\theta_1)$$

$$|z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_2\sin\theta_1)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos^2\theta_1 - 2r_1r_2\sin^2\theta_1}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\sin^2\theta_1 + \cos^2\theta_1)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2} = \sqrt{(r_1 - r_2)^2}$$

$$\Rightarrow |z_1 - z_2| = r_1 - r_2 \quad [\because r = |z|]$$

$$= |z_1| - |z_2|$$

Hence proved.

Q. 21 Solve the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$.

Sol. Given that, $\text{Re}(z^2) = 0, |z| = 2$

Let $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

$\because \sqrt{x^2 + y^2} = 2$

$\Rightarrow x^2 + y^2 = 4 \quad \text{...(i)}$

and $\text{Re}(z) = x$

Also, $z = x + iy$

$\Rightarrow z^2 = x^2 + 2ixy - y^2$

$\Rightarrow z^2 = (x^2 - y^2) + 2ixy$

$\Rightarrow \text{Re}(z^2) = x^2 - y^2 \quad [\because \text{Re}(z^2) = 0]$

$\Rightarrow x^2 - y^2 = 0 \quad \text{...(ii)}$

From Eqs. (i) and (ii),

$$x^2 + x^2 = 4$$

$\Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

$\therefore y = \pm\sqrt{2}$

$\because z = x + iy$

$\Rightarrow z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$

Q. 22 Find the complex number satisfying the equation $z + \sqrt{2}\,|(z+1)| + i = 0$.

Sol. Given equation is $z + \sqrt{2}\,|(z+1)| + i = 0$ ...(i)

Let $z = x + iy$

$\Rightarrow x + iy + \sqrt{2}\,|x + iy + 1| + i = 0$

$\Rightarrow x + i(1+y) + \sqrt{2}\left[\sqrt{(x+1)^2 + y^2}\right] = 0$

$\Rightarrow x + i\,(1+y) + \sqrt{2}\,\sqrt{(x^2 + 2x + 1 + y^2)} = 0$

$\Rightarrow x + \sqrt{2}\,\sqrt{x^2 + 2x + 1 + y^2} = 0$

$\Rightarrow x^2 = 2(x^2 + 2x + 1 + y^2$

$\Rightarrow x^2 + 4x + 2y^2 + 2 = 0$ ...(ii)

$1 + y = 0$

$\Rightarrow y = -1$

For $y = -1$, $x^2 + 4x + 2 + 2 = 0$ [using Eq. (ii)]

$\Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0$

$\Rightarrow x + 2 = 0 \Rightarrow x = -2$

$\therefore z = x + iy = -2 - i$

Q. 23 Write the complex number $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ in polar form.

Sol. Given that, $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}} = \dfrac{-\sqrt{2}\left[\dfrac{-1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right]}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$

$= \dfrac{-\sqrt{2}[\cos(\pi - \pi/4) + i\sin(\pi - \pi/4)]}{\cos\pi/3 + i\sin\pi/3}$

$= \dfrac{-\sqrt{2}[\cos 3\pi/4 + i\sin 3\pi/4]}{\cos\pi/3 + i\sin\pi/3}$

$= -\sqrt{2}\left[\cos\left(\dfrac{3\pi}{4} - \dfrac{\pi}{3}\right) + i\sin\left(\dfrac{3\pi}{4} - \dfrac{\pi}{3}\right)\right]$

$= -\sqrt{2}\left[\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right]$

Q. 24 If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$, then show that $\bar{z}\,w = -i$.

Sol. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

Also, $|zw| = |z|\,|w| = r_1\,r_2 = 1$

$\therefore r_1 r_2 = 1$

Further, $\arg(z) = \theta_1$ and $\arg(w) = \theta_2$

But $\arg(z) - \arg(w) = \frac{\pi}{2}$

$\Rightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$

$\Rightarrow \quad \arg\left(\frac{z}{w}\right) = \frac{\pi}{2}$

Now, to prove $\bar{z}w = -i$

$$\begin{aligned} \text{LHS} &= \bar{z}w \\ &= r_1(\cos\theta_1 - i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)] \\ &= r_1 r_2[\cos(-\pi/2) + i\sin(-\pi/2)] \\ &= 1[0 - i] \\ &= -i = \text{RHS} \end{aligned}$$

Hence proved.

Q. 25 Fill in the blanks of the following.

(i) For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \cdots$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ...

(iii) The number $\frac{(1-i)^3}{1-i^3}$ is equal to ...

(iv) The sum of the series $i + i^2 + i^3 + \cdots$ upto 1000 terms is ...

(v) Multiplicative inverse of $1 + i$ is ...

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 = \cdots$

(vii) $\arg(z) + \arg\bar{z}$ where, $(\bar{z} \neq 0)$ is ...

(viii) If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are ... and ...

(ix) If $\left|\frac{z-2}{z+2}\right| = \frac{\pi}{6}$, then the locus of z is ...

(x) If $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then $z = \cdots$

Sol. (i) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$$\begin{aligned} &= |az_1|^2 + |bz_2|^2 - 2\text{Re}(az_1 \cdot b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2\text{Re}(az_1 \cdot b\bar{z}_2) \\ &= (a^2 + b^2)|z_1|^2 + (a^2 + b^2)|z_2|^2 \\ &= (a^2 + b^2)(|z_1|^2 + |z_2|^2) \end{aligned}$$

(ii) $\sqrt{-25} \times \sqrt{-9} = i\sqrt{25} \times i\sqrt{9} = i^2(5 \times 3) = -15$

(iii) $\frac{(1-i)^3}{1-i^3} = \frac{(1-i)^3}{(1-i)(1+i+i^2)}$

$$= \frac{(1-i)^2}{i} = \frac{1+i^2-2i}{i} = \frac{-2i}{i} = -2$$

(iv) $i + i^2 + i^3 + \ldots$ upto 1000 terms $= i + i^2 + i^3 + i^4 + \ldots i^{1000} = 0$

$$\left[\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{1000} i^n = 0\right]$$

(v) Multiplicative inverse of $1 + i = \dfrac{1}{1+i} = \dfrac{1-i}{1-i^2} = \dfrac{1}{2}(1-i)$

(vi) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, which is real.

If $z_1 + z_2$ is real, then $y_1 + y_2 = 0$

$\Rightarrow \quad y_1 = -y_2$

$\because \quad z_2 = x_2 - iy_1$

$\Rightarrow \quad z_2 = \bar{z}_1$ [when $x_1 = x_2$]

(vii) $\arg(z) + \arg(\bar{z}), (\bar{z} \neq 0)$

$\Rightarrow \quad \theta + (-\theta) = 0$

(viii) Given that, $|z + 4| \leq 3$

For the greatest value of $|z + 1|$.

$$\Rightarrow \quad |z+1| = |z + 4 - 3| \leq |z + 4| + |-3|$$

$$= |z + 4 - 3| \leq 3 + 3$$

$$= |z + 4 - 3| \leq 6$$

So, greatest value of $|z+1| = 6$

For, now, least value of $|z+1|$, we know that the least value of the modulus of a complex number is zero. So, the least value of $|z+1|$ is zero.

(ix) Given that, $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$

$$\Rightarrow \quad \frac{|x + iy - 2|}{|x + iy + 2|} = \frac{\pi}{6} \Rightarrow \frac{|x - 2 + iy|}{|x + 2 + iy|} = \frac{\pi}{6}$$

$$\Rightarrow \quad 6|x - 2 + iy| = \pi\,|x + 2 + iy|$$

$$\Rightarrow \quad 6\sqrt{(x-2)^2 + y^2} = \pi\sqrt{(x+2)^2 + y^2}$$

$$\Rightarrow \quad 36[x^2 + 4 - 4x + y^2] = \pi^2[x^2 + 4x + 4 + y^2]$$

$\Rightarrow (36 - \pi^2)x^2 + (36 - \pi^2)y^2 - (144 + 4\pi^2)x + 144 + 4\pi^2 = 0$, which is a circle.

(x) Given that, $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$

Let $\quad z = x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad |z| = r = 4$ and $\arg(z) = \theta$

$\because \quad \tan\theta = \dfrac{5\pi}{6}$

$$\Rightarrow \quad z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 4[\cos(\pi - \pi/6) + i\sin(\pi - \pi/6)]$$

$$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right] = -2\sqrt{3} + 2i$$

True/False

Q. 26 State true or false for the following.

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z, the minimum value of $|z| + |z-1|$ is 1.

(iv) The locus represented by $|z-1| = |z-i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).

(v) If z is a complex number such that $z \neq 0$ and $\text{Re}(z) = 0$, then, $\text{Im}(z^2) = 0$.

(vi) The inequality $|z-4| < |z-2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

Sol. (i) ***False***

We can compare two complex numbers when they are purely real. Otherwise comparison of complex number is not possible.

(ii) ***False***

$$(x, y)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \text{ which is false.}$$

(iii) ***True***

Let $z = x + iy$

$$|z| + |z-1| = \sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2}$$

If $x = 0$, $y = 0$, then the value of $|z| + |z-1| = 1$.

(iv) ***True***

Let $z = x + iy$

$$|z-1| = |z-i|$$

Then,
$$|x - 1 + iy| = |x - i(1-y)|$$
$$(x-1)^2 + y^2 = x^2 + (1-y)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$$
$$-2x + 1 = 1 - 2y$$
$$-2x + 2y = 0$$
$$x - y = 0 \quad \text{...(i)}$$

Equation of a line through the points (1, 0) and (0, 1),

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow \quad y = -(x-1) \Rightarrow x + y = 1 \quad \text{...(ii)}$$

which is perpendicular to the line $x - y = 0$.

(v) ***False***

Let $z = x + iy$, $z \neq 0$ and Re $(z) = 0$

i.e., $x = 0$

$\therefore$ $z = iy$

$\text{Im}(z^2) = i^2 y^2 = -y^2$ which is real.

(vi) ***True***

Given inequality, $|z-4| < |z-2|$

Let $z = x + iy$

$\therefore$ $|x - 4 + iy| < |x - 2 + iy|$

$\Rightarrow$ $\sqrt{(x-4)^2 + y^2} < \sqrt{(x-2)^2 + y^2}$

$\Rightarrow$ $(x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow$ $x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$

$\Rightarrow$ $-8x + 16 < -4x + 4$

$\Rightarrow$ $-8x < -4x - 12$

$\Rightarrow$ $-4x < -12$

$\Rightarrow$ $4x > 12$

$\Rightarrow$ $x > 3$

(vii) ***False***

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$$|x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1| + |x_2 + iy_2|$$

$\Rightarrow$ $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$\Rightarrow$ $x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow$ $2x_1x_2 + 2y_1y_2 = 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow$ $x_1x_2 + y_1y_2 = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 = x_1^2x_2^2 + y_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2$$

$\Rightarrow$ $(x_1y_2 - x_2y_1)^2 = 0$

$\Rightarrow$ $x_1y_2 = x_2y_1$

$\Rightarrow$ $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$

$\Rightarrow$ $\left(\dfrac{y_1}{x_1}\right) - \left(\dfrac{y_2}{x_2}\right) = 0$

$\Rightarrow$ $\arg(z_1) - \arg(z_2) = 0$

(viii) ***True***

We know that, 2 is a real number.

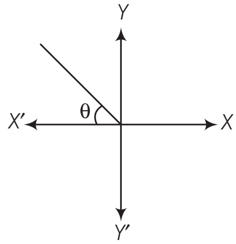
Since, 2 is not a complex number.

Q. 27 Match the statements of Column A and Column B.

	Column A		Column B
(i)	The polar form of $i+\sqrt{3}$ is	(a)	Perpendicular bisector of segment joining (–2, 0) and (2, 0).
(ii)	The amplitude of $-1+\sqrt{-3}$ is	(b)	On or outside the circle having centre at (0, – 4) and radius 3.
(iii)	It $\lvert z+2\rvert=\lvert z-2\rvert$, then locus of z is	(c)	$\frac{2\pi}{3}$
(iv)	It $\lvert z+2i\rvert=\lvert z-2i\rvert$, then locus of z is	(d)	Perpendicular bisectar of segment joining (0, – 2) and (0, 2).
(v)	Region represented by	(e)	$2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
(vi)	Region represented by $\lvert z+4\rvert\le 3$ is	(f)	On or inside the circle having centre (– 4, 0) and radius 3 units.
(vii)	Conjugate of $\frac{1+2i}{1-i}$ lies in	(g)	First quadrant
(viii)	Reciprocal of $1-i$ lies in	(h)	Third quadrant

Sol. **(i)** Given,
$$z=i+\sqrt{3}=r(\cos\theta+\sin\theta)$$
$\because$ $$r\cos\theta=\sqrt{3},\ r\sin\theta=1$$
$\Rightarrow$ $$r^2=1+3=4\ \Rightarrow\ r=2 \qquad [\because r>0]$$
$\Rightarrow$ $$\tan\alpha=\left|\frac{r\sin\theta}{r\cos\theta}\right|=\frac{1}{\sqrt{3}}$$
$\Rightarrow$ $$\tan\alpha=\frac{1}{\sqrt{3}}\ \Rightarrow\ \alpha=\frac{\pi}{6}$$
$\because$ $$x>0,\ \ y>0$$
and $$\arg(z)=\theta=\frac{\pi}{6}$$
So the polar form of z is $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$.

(ii) Given that,
$$z=-1+\sqrt{-3}=-1+i\sqrt{3}$$
$\therefore$ $$\tan\alpha=\left|\frac{\sqrt{3}}{-1}\right|=\sqrt{3}$$
$\Rightarrow$ $$\tan\alpha=\tan\frac{\pi}{3}\ \Rightarrow\ \alpha=\frac{\pi}{3}$$



$\because$ $$x<0,\ \ y>0$$
$\Rightarrow$ $$\theta=\pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

(iii) Given that, $|z+2| = |z-2|$

$\Rightarrow \quad |x+2+iy| = |x-2+iy|$

$\Rightarrow \quad (x+2)^2 + y^2 = (x-2)^2 + y^2$

$\Rightarrow \quad x^2 + 4x + 4 = x^2 - 4x + 4 \quad \Rightarrow \quad 8x = 0$

$\therefore \quad x = 0$

It is a straight line which is a perpendicular bisector of segment joining the points (–2, 0) and (2, 0).

(iv) Given that, $|z+2i| = |z-2i|$

$\Rightarrow \quad |x+i(y+2)| = |x+i(y-2)|$

$\Rightarrow \quad x^2 + (y+2)^2 = x^2 + (y-2)^2$

$\Rightarrow \quad 4y = 0 \quad \Rightarrow \quad y = 0$

It is a straight line, which is a perpendicular bisector of segment joining (0, –2) and (0, 2).

(v) Given that, $|z+4i| \geq 3 = |x+iy+4i| \geq 3$

$\Rightarrow \quad = |x+i(y+4)| \geq 3$

$\Rightarrow \quad = \sqrt{x^2 + (y+4)^2} \geq 3$

$\Rightarrow \quad = x^2 + (y+4)^2 \geq 9$

$\Rightarrow \quad = x^2 + y^2 + 8y + 16 \geq 9$

$= x^2 + y^2 + 8y + 7 \geq 0$

Which represent a circle. On or outside having centre (0, – 4) and radius 3.

(vi) Given that, $|z+4| \leq 3$

$\Rightarrow \quad |x+iy+4| \leq 3$

$\Rightarrow \quad |x+4+iy| \leq 3$

$\Rightarrow \quad \sqrt{(x+4)^2 + y^2} \leq 3$

$\Rightarrow \quad (x+4)^2 + y^2 \leq 9$

$\Rightarrow \quad x^2 + 8x + 16 + y^2 \leq 9$

$\Rightarrow \quad x^2 + 8x + y^2 + 7 \leq 0$

It represent the region which is on or inside the circle having the centre (– 4, 0) and radius 3.

(vii) Given that, $z = \dfrac{1+2i}{1-i} = \dfrac{(1+2i)(1+i)}{(1-i)(1+i)}$

$= \dfrac{1+2i+i+2i^2}{1-i^2} = \dfrac{1-2+3i}{1+1} = \dfrac{-1+3i}{2}$

$\therefore \quad \bar{z} = \dfrac{-1}{2} - \dfrac{3i}{2}$

Hence, $\left(\dfrac{-1}{2}, \dfrac{-3}{2}\right)$ lies in third quadrant.

(viii) Given that, $z = 1-i$

$\therefore \quad \dfrac{1}{z} = \dfrac{1}{1-i} = \dfrac{1+i}{(1-i)(1+i)} = \dfrac{1+i}{1-i^2} = \dfrac{1}{2}(1+i)$

Hence, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ lies in first quadrant.

Hence, the correct matches are **(a)** → (v), **(b)** → (iii), **(c)** → (i), **(d)** → (iv), **(e)** → (ii), **(f)** → (vi), **(g)** → (viii), **(h)** → (vii)

Q. 28 What is the conjugate of $\dfrac{2-i}{(1-2i)^2}$?

Sol. Given that,

$$z = \frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i}$$
$$= \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i}$$
$$= \frac{(2-i)}{-(3+4i)} = -\left[\frac{(2-i)(3-4i)}{(3+4i)(3-4i)}\right]$$
$$= -\left(\frac{6-8i-3i+4i^2}{9+16}\right) = -\frac{(-11i+2)}{25}$$
$$= \frac{-1}{25}(2-11i) \Rightarrow z = \frac{1}{25}(-2+11i)$$

$\therefore$
$$\bar{z} = \frac{1}{25}(-2-11i) = \frac{-2}{25} - \frac{11}{25}i$$

Q. 29 If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$.

Sol. Given that, $|z_1| = |z_2|$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\Rightarrow$ $|x_1 + iy_1| = |x_2 + iy_2|$

$\Rightarrow$ $x_1^2 + y_1^2 = x_2^2 + y_2^2$

$\Rightarrow$ $x_1^2 = x_2^2,\ y_1^2 = y_2^2$

$\Rightarrow$ $x_1 = \pm x_2,\ y_1 = \pm y_2$

$\Rightarrow$ $z_1 = x_1 + iy_1$ or $z_1 = \pm x_2 \pm iy_2$

Hence, it is not neccessary that $z_1 = z_2$.

Q. 30 If $\dfrac{(a^2+1)^2}{2a-i} = x + iy$, then what is the value of $x^2 + y^2$?

Sol. Given that,
$$\frac{(a^2+1)^2}{2a-i} = x+iy \Rightarrow \frac{(a^2+1)^2}{(2a-i)} = x+iy$$

$$\Rightarrow \frac{(a^2+1)^2(2a+i)}{(2a-i)(2a+i)} = x+iy$$

$$\Rightarrow \frac{(a^2+1)^2(2a+i)}{4a^2+1} = x+iy$$

$$\Rightarrow x = \frac{2a(a^2+1)^2}{4a^2+1} \quad \text{and} \quad y = \frac{(a^2+1)^2}{4a^2+1}$$

$\therefore$
$$x^2+y^2 = 4a^2\left[\frac{(a^2+1)^2}{4a^2+1}\right]^2 + \left[\frac{(a^2+1)^2}{4a^2+1}\right]^2$$
$$= \frac{(4a^2+1)(a^2+1)^4}{(4a^2+1)^2} = \frac{(a^2+1)^4}{(4a^2+1)}$$

Q. 31 Find the value of z, if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

Sol. Let $z = r(\cos\theta + i\sin\theta)$

Also, $|z| = r = 4$ and $\theta = \frac{5\pi}{6}$ $[\because \arg(z) = \theta]$

$$\therefore \quad z = 4\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] \quad [\because z = r(\cos\theta + i\sin\theta)]$$

$$= 4\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 4\left[-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = -2\sqrt{3} + 2i$$

Q. 32 Find the value of $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$.

Thinking Process

First, convert the given expression in the formed $a + ib$, then use $|a + ib| = \sqrt{a^2 + b^2}$.

Sol. Given that, $\left|(1+i)\frac{(2+i)}{(3+i)}\right| = \left|\frac{(2+i+2i+i^2)}{(3+i)}\right| = \left|\frac{2+3i-1}{3+i}\right|$

$$= \left|\frac{1+3i}{3+i}\right| = \left|\frac{(1+3i)(3-i)}{(3+i)(3-i)}\right|$$

$$= \left|\frac{3+9i-i-3i^2}{9-i^2}\right| = \left|\frac{3+8i+3}{9+1}\right| = \left|\frac{6+8i}{10}\right|$$

$$= \sqrt{\frac{6^2}{100} + \frac{8^2}{100}} = \sqrt{\frac{36+64}{100}} = \sqrt{\frac{100}{100}} = 1$$

Q. 33 Find the principal argument of $(1 + i\sqrt{3})^2$.

Thinking Process

Let $z = a + ib$, then the polar form of z is $r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$. Here, θ is argument or amplitude of z i.e., arg $(z) = \theta$. The principal argument is a unique value of θ such that $-\pi \le \theta \le \pi$.

Sol. Given that, $z = (1 + i\sqrt{3})^2$

$\Rightarrow$ $z = 1 - 3 + 2i\sqrt{3} \Rightarrow z = -2 + i2\sqrt{3}$

$\Rightarrow$ $\tan\alpha = \left|\frac{2\sqrt{3}}{-2}\right| = |-\sqrt{3}| = \sqrt{3}$ $\left[\because \tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|\right]$

$\Rightarrow$ $\tan\alpha = \tan\frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$

$\because$ $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$

$\Rightarrow$ $\arg(z) = \pi - \frac{\pi}{3} \Rightarrow = \frac{2\pi}{3}$

Q. 34 Where does z lie, if $\left|\frac{z-5i}{z+5i}\right| = 1$?

Thinking Process

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$.

Also, use the modulus property i.e., $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$,

Sol. Let $z = x + iy$

Given that, $\left|\frac{z-5i}{z+5i}\right| = \left|\frac{x+iy-5i}{x+iy+5i}\right|$

$\Rightarrow \left|\frac{z-5i}{z+5i}\right| = \frac{|x+i(y-5)|}{|x+i(y+5)|}$ $\left[\because \left|\frac{z-5i}{z+5i}\right| = 1\right]$

$\Rightarrow \left|\frac{z-5i}{z+5i}\right| = \frac{\sqrt{x^2+(y-5)^2}}{\sqrt{x^2+(y+5)^2}}$

On squaring both sides, we get

$$x^2 + (y-5)^2 = x^2 + (y+5)^2$$

$\Rightarrow -10y = +10y$

$\Rightarrow 20y = 0$

$\therefore y = 0$

So, z lies on real axis.

Objective Type Questions

Q. 35 $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(a) $x = n\pi$

(b) $x = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$

(c) $x = 0$

(d) No value of x

Sol. *(d)* Let $z = \sin x + i\cos 2x$

and $\bar{z} = \sin x - i\cos 2x$ ...(i)

Given that, $\bar{z} = \cos x - i\sin 2x$ ...(ii)

$\therefore \sin x - i\cos 2x = \cos x - i\sin 2x$

$\Rightarrow \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow \tan x = \tan\frac{\pi}{4}$ and $\tan 2x = \tan\frac{\pi}{4}$

$\Rightarrow x = n\pi + \frac{\pi}{4}$ and $2x = n\pi + \frac{\pi}{4}$

$\Rightarrow 2x - x = 0 \Rightarrow x = 0$

Q. 36 The real value of α for which the expression $\dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real is

(a) $(n+1)\dfrac{\pi}{2}$ (b) $(2n+1)\dfrac{\pi}{2}$ (c) $n\pi$ (d) None of these

where, $n \in N$

Thinking Process

First, convert the given expansion into $a + ib$ form and then check whether the complex number $a + ib$ is purely real.

Sol. ***(c)*** Given expression, $z = \dfrac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ [let]

$$= \frac{(1 - i\sin\alpha)(1 - 2i\sin\alpha)}{(1 + 2i\sin\alpha)(1 - 2i\sin\alpha)}$$

$$= \frac{1 - i\sin\alpha - 2i\sin\alpha + 2i^2\sin^2\alpha}{1 - 4i^2\sin^2\alpha}$$

$$= \frac{1 - 3i\sin\alpha - 2\sin^2\alpha}{1 + 4\sin^2\alpha}$$

$$= \frac{1 - 2\sin^2\alpha}{1 + 4\sin^2\alpha} - \frac{3i\sin\alpha}{1 + 4\sin^2\alpha}$$

It is given that z is a purely real.

$$\therefore \quad \frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0$$

$$\Rightarrow \quad -3\sin\alpha = 0 \Rightarrow \sin\alpha = 0$$

$$\alpha = n\pi$$

Q. 37 If $z = x + iy$ lies in the third quadrant, then $\dfrac{\bar{z}}{z}$ also lies in the third quadrant, if

(a) $x > y > 0$ (b) $x < y < 0$ (c) $y < x < 0$ (d) $y > x > 0$

Sol. ***(b)*** Given that, $z = x + iy$ lies in third quadrant.

$$x < 0 \text{ and } y < 0.$$

Now, $$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$$

$$\frac{\bar{z}}{z} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$$

Since, $\dfrac{\bar{z}}{z}$ also lies in third quadrant.

$$\therefore \quad \frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0$$

$$x^2 - y^2 < 0 \text{ and } -2xy < 0$$

$$\Rightarrow \quad x^2 < y^2 \text{ and } xy > 0$$

So, $$x < y < 0$$

Q. 38 The value of $(z+3)(\bar{z}+3)$ is equivalent to

(a) $|z+3|^2$ (b) $|z-3|$ (c) z^2+3 (d) None of these

Sol. *(a)* Given that, $(z+3)(\bar{z}+3)$

Let $z = x + iy$

$$\Rightarrow (z+3)(\bar{z}+3) = (x+iy+3)(x+3-iy)$$
$$= (x+3)^2 - (iy)^2 = (x+3)^2 + y^2$$
$$= |x+3+iy|^2 = |z+3|^2$$

Q. 39 If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(a) $x = 2n+1$ (b) $x = 4n$ (c) $x = 2n$ (d) $x = 4n+1$

where, $n \in N$

Sol. *(b)* Given that, $\left(\frac{1+i}{1-i}\right)^x = 1$

$$\Rightarrow \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x = 1 \Rightarrow \left[\frac{1+2i+i^2}{1-i^2}\right]^x = 1$$

$$\Rightarrow \left[\frac{2i}{1+1}\right]^x = 1 \Rightarrow \left[\frac{2i}{2}\right]^x = 1$$

$$\Rightarrow i^x = 1 \Rightarrow i^x = (i^{4n}) \quad [\because i^{4n} = 1, n \in N]$$

$$\Rightarrow x = 4n$$

Q. 40 A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta \ (\alpha, \beta \in R)$, if $\alpha^2 + \beta^2$ is equal to

(a) 1 (b) −1 (c) 2 (d) − 2

Sol. *(a)* Given equation, $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta \ (\alpha, \beta \in R)$

$$\Rightarrow \left[\frac{3-4ix}{3+4ix}\right] = \alpha - i\beta$$

Now, $$(\alpha - i\beta) = \frac{(3-4ix)(3-4ix)}{(3+4ix)(3-4ix)} = \frac{9+16i^2x^2-24ix}{9-16i^2x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9-16x^2-24ix}{9+16x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9-16x^2}{9+16x^2} - \frac{i24x}{9+16x^2} \quad \ldots(i)$$

$$\therefore \alpha + i\beta = \frac{9-16x^2}{9+16x^2} + \frac{i24x}{9+16x^2} \quad \ldots(ii)$$

So, $(\alpha - i\beta)(\alpha + i\beta) = \left(\frac{9-16x^2}{9+16x^2}\right)^2 - \left(\frac{i24x}{9+16x^2}\right)^2$

$$\therefore \quad \alpha^2 + \beta^2 = \frac{81 + 256x^4 - 288x^2 + 576x^2}{(9+16x^2)^2}$$

$$= \frac{81 + 256x^4 + 288x^2}{(9+16x^2)^2}$$

$$= \frac{(9+16x^2)^2}{(9+16x^2)^2} = 1$$

Q. 41 Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2| = |z_1||z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1) \cdot \arg(z_2)$

(c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \ge |z_1| - |z_2|$

Sol. *(a)* Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow |z_1| = r_1$ ...(i)

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow |z_2| = r_2$ ...(ii)

Now, $z_1 z_2 = r_1 r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2]$

$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\Rightarrow |z_1 z_2| = r_1 r_2$

$\therefore |z_1 z_2| = |z_1||z_2|$ [using Eqs. (i) and (ii)]

Q. 42 The point represented by the complex number $(2 - i)$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

Thinking Process

Here, $z < i\alpha$ is a complex number, where modulus is r and argument $(\theta + \alpha)$. If P (z) rotates in clockwise sense through an angle α, then its new position will be $z(\theta - i\alpha)$.

Sol. *(b)* Given that, $z = 2 - i$

It is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction

$\therefore$ New position $= ze^{-i\pi/2} = (2-i)e^{-i\pi/2}$

$$= (2-i)\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right] = (2-i)[0-i]$$

$$= -2i - 1 = -1 - 2i$$

Q. 43 If $x, y \in R$, then $x + iy$ is a non-real complex number, if

(a) $x = 0$ (b) $y = 0$ (c) $x \neq 0$ (d) $y \neq 0$

Sol. *(d)* Given that, $x, y \in R$

Then, $x + iy$ is non-real complex number if and only if $y \neq 0$.

Q. 44 If $a+ib=c+id$, then

(a) $a^2+c^2=0$ (b) $b^2+c^2=0$

(c) $b^2+d^2=0$ (d) $a^2+b^2=c^2+d^2$

Thinking Process

If two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are equal, then

$$|z_1|=|z_2| \Rightarrow \sqrt{x_1^2+y_1^2}=\sqrt{x_2^2+y_2^2}$$

Sol. *(d)* Given that, $a+ib=c+id$

$\Rightarrow$ $|a+ib|=|c+id|$

$\Rightarrow$ $\sqrt{a^2+b^2}=\sqrt{c^2+d^2}$

On squaring both sides, we get

$$a^2+b^2=c^2+d^2$$

Q. 45 The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right|=1$ lies on

(a) circle $x^2+y^2=1$ (b) the X-axis

(c) the Y-axis (d) the line $x+y=1$

Sol. *(b)* Given that, $\left|\frac{i+z}{i-z}\right|=1$

Let $z=x+iy$

$\therefore$ $\left|\frac{x+i(y+1)}{-x-i(y-1)}\right|=1 \Rightarrow \frac{x^2+(y+1)^2}{x^2+(y-1)^2}=1$

$\Rightarrow$ $x^2+(y+1)^2=x^2+(y-1)^2$

$\Rightarrow$ $4y=0 \Rightarrow y=0$

So, z lies on X-axis (real axis).

Q. 46 If z is a complex number, then

(a) $|z^2|>|z|$ (b) $|z^2|=|z|^2$ (c) $|z^2|<|z|^2$ (d) $|z^2|\geq|z|^2$

Sol. *(b)* If z is a complex number, then $z=x+iy$

$$|z|=|x+iy| \text{ and } |z|^2=|x+iy|^2$$

$\Rightarrow$ $|z|^2=x^2+y^2$ ...(i)

and $z^2=(x+iy)^2=x^2+i^2y^2+i2xy$

$z^2=x^2-y^2+i2xy$

$\Rightarrow$ $|z^2|=\sqrt{(x^2-y^2)^2+(2xy)^2}$

$\Rightarrow$ $|z^2|=\sqrt{x^4+y^4-2x^2y^2+4x^2y^2}$

$\Rightarrow$ $|z^2|=\sqrt{x^4+y^4+2x^2y^2}=\sqrt{(x^2+y^2)^2}$

$\Rightarrow$ $|z^2|=x^2+y^2$ ...(ii)

From Eqs. (i) and (ii),

$$|z|^2=|z^2|$$

Q. 47 $|z_1 + z_2| = |z_1| + |z_2|$ is possible, if

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

Sol. *(c)* Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$$\Rightarrow |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)| = |r_1(\cos\theta_1 + i\sin\theta_1)| + |r_2(\cos\theta_2 + i\sin\theta_2)|$$

$$\Rightarrow |(r_1\cos\theta_1 + r_2\cos\theta) + i(r_1\sin\theta_1 + r_2\sin\theta_2)| = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2} = r_1 + r_2$$

$$\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1r_2[\cos(\theta_1 - \theta_2)]} = r_1 + r_2$$

On squaring both sides, we get

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2$$

$$\Rightarrow 2r_1r_2[1 - \cos(\theta_1 - \theta_2)] = 0$$

$$\Rightarrow 1 - \cos(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = \cos 0°$$

$$\Rightarrow \theta_1 - \theta_2 = 0°$$

$$\Rightarrow \theta_1 = \theta_2$$

$$\therefore \arg(z_1) = \arg(z_2)$$

Q. 48 The real value of θ for which the expression $\frac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi + (-1)^n\frac{\pi}{4}$

(c) $2n\pi \pm \frac{\pi}{2}$ (d) None of these

Sol. *(c)* Given expression $= \frac{1 + i\cos\theta}{1 - 2i\cos\theta} = \frac{(1 + i\cos\theta)(1 + 2i\cos\theta)}{(1 - 2i\cos\theta)(1 + 2i\cos\theta)}$

$$= \frac{1 + i\cos\theta + 2i\cos\theta + 2i^2\cos^2\theta}{1 - 4i^2\cos^2\theta}$$

$$= \frac{1 + 3i\cos\theta - 2\cos^2\theta}{1 + 4\cos^2\theta}$$

For real value of θ, $\frac{3\cos\theta}{1 + 4\cos^2\theta} = 0$

$$\Rightarrow 3\cos\theta = 0$$

$$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$$

$$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2}$$

Q. 49 The value of arg (x), when $x < 0$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) None of these

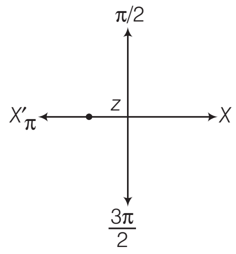
Sol. ***(c)*** Let

$$z = x + 0i \text{ and } x < 0$$

$$|z| = \sqrt{(-1)^2 + (0^2)} = 1$$

Since, the point $(x, 0)$ represent $z = x + 0i$ lies on the negative side of real axis.

$\therefore$ Principal arg $(z) = \pi$



Q. 50 If $f(z) = \frac{7 - z}{1 - z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is equal to

(a) $\frac{|z|}{2}$ (b) $|z|$

(c) $2|z|$ (d) None of these

Sol. ***(a)*** Let

$$z = 1 + 2i$$

$\Rightarrow$

$$|z| = \sqrt{1 + 4} = \sqrt{5}$$

Now,

$$f(z) = \frac{7 - z}{1 - z^2} = \frac{7 - 1 - 2i}{1 - (1 + 2i)^2}$$

$$= \frac{6 - 2i}{1 - 1 - 4i^2 - 4i} = \frac{6 - 2i}{4 - 4i}$$

$$= \frac{(3 - i)(2 + 2i)}{(2 - 2i)(2 + 2i)}$$

$$= \frac{6 - 2i + 6i - 2i^2}{4 - 4i^2} = \frac{6 + 4i + 2}{4 + 4}$$

$$= \frac{8 + 4i}{8} = 1 + \frac{1}{2}i$$

$$f(z) = 1 + \frac{1}{2}i$$

$\therefore$

$$|f(z)| = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{4 + 1}{4}} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$$